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The lower bound on the minimal degree of the matrices in first-order relativistic wave equations

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Abstract. A proof is given, on the basis of a theorem due to Gårding, for a lower bound on the degree $(l+2)$ of the minimal equation of matrices β^μ in first-order unique-mass relativistic wave equations. It is not necessary, for the applicability of the theorem, that a hermitising operator should exist or that the equation be irreducible; and the generalisation of the bound to multimass equations is also straightforward. The bound is not, in general, linked to the physical spin or spins s allowed by the wave equation or the maximum spin j_m contained in the wavefunction. However, in the physically important case of irreducible equations which admit a hermitising operator, the bound becomes $(l+2) \geqslant (2j_m+1)$, which is stronger than the bound $(2s+1)$ suggested in the recent literature.

1. Introduction

It is well known (Harish-Chandra 1947) that if the first-order relativistic wave equation

$$(\mathrm{i}\beta^\mu \partial_\mu - m) = 0 \qquad (1)$$

is to describe particles of unique mass m, the minimal equation of the matrix β^0 must have the form

$$(\beta^0)^{l+2} = (\beta^0)^l. \qquad (2)$$

More generally, the β^μ must be such that

$$\sum_{\mathscr{P}} (\beta^{\mu_1}\beta^{\mu_2} - g^{\mu_1\mu_2})\beta^{\mu_3} \ldots \beta^{\mu_{l+2}} = 0 \qquad (3)$$

where the sum is over all permutations $\mathscr{P}$ of the indices $\mu_1, \mu_2, \ldots, \mu_{l+2}$.

Also well known is the work of Umezawa and Visconti (1956) (see also Umezawa 1956 and Takahashi 1969) wherein it was concluded that $l = 2s-1$ (s being the physical spin of the particle described by the wave equation). This conclusion was based on arguments which seemed to lead to coincident upper and lower bounds on l, namely $l \leqslant 2s-1$ and $l \geqslant 2s-1$. It had been tacitly assumed by these authors that the physical spin s was also the maximum spin j_m appearing in the transformation property of ψ. Glass (1971a) noted that in the more general situation where $j_m > s$, their argument would have given $l \leqslant 2j_m - 1$ (instead of $2s-1$), but even this modified bound would not be valid in general, there being a lacuna in the original proof thereof. In fact the physical spin has little to do with any upper bound on l, as has been noted by Mathews *et al* (1980). It is the number, multiplicities and connectivities of the Lorentz irreducible representations (IR) occurring in the transformation property of the wavefunction which go into the determination of l. From a knowledge of these, one readily obtains an absolute upper bound on l in terms of the *size* of the largest of the spin blocks into which β^0 can be decomposed; if the size of the spin-j block is L_j, then

$$l \leqslant \max(L_j - 2\,\delta_{js}) \tag{4}$$

in any unique-mass unique-spin theory without degeneracy. A better bound (which has already been used in the context of the Singh–Hagen (1974) integer spin equations in Mathews *et al* (1980)) is

$$l \leqslant \max(r_j + 1 - 2\,\delta_{js}) \tag{5}$$

where r_j is the rank† of the spin-j block. Even this might be improved upon under particular conditions or in particular cases: in half-integer spin theories with parity invariance, for instance, l cannot exceed $\frac{1}{2}\max(r_j + 1 - 2\,\delta_{js})$.

While the position regarding the upper bound on l has thus been clarified considerably, the same cannot be said of the question as to what factors determine how small l may be. It was suggested by Chandrasekharan *et al* (1972) that in theories where $s < j_m$, the UV bounds should be interpreted as $(2s-1) \leqslant l \leqslant (2j_m - 1)$. As already noted, the upper bound is not generally valid, and it has been pointed out by Mathews *et al* (1980) that the above lower bound too is not valid in general. In a recent paper Cox (1981) has argued that in theories in which a hermitising operator η is defined and a Klein–Gordon divisor exists as a polynomial in $\beta \cdot \partial$ with the β^μ forming a set irreducible under the proper Lorentz group, the lower bound $l \geqslant (2s-1)$ should remain. The argument is rather heuristic and rests on the need for having at least a minimum number of Lorentz IR to form an unbroken chain connecting any IR containing the physical spin to its conjugate. The bound is indeed honoured under the assumed conditions, but no analysis of the precise roles of various conditions in the determination of a bound has however been made in that paper or elsewhere in the literature. Our aim in this paper is to pinpoint the factors which determine the lower bound in Lorentz invariant theories (whether or not they admit any η and whether or not the β^μ form an irreducible set). It will be seen that the physical spin s does not enter into the proof of the bound we establish. In general, the absolute lower bound on l is given by $(2k-1)$, where k is the maximum rank of symmetric tensors which can be constructed from the matrix four-vector β^μ; and the value of k in turn can be inferred from a result due to Glass (1971b), based on a fundamental theorem of Gårding (1944). This theorem and its implications for the present problem will be dealt with in § 3, while in § 2 we point out a flaw in the Umezawa–Visconti (1956) proof of the lower bound, namely their premise regarding the transformation property of the Klein–Gordon divisor‡ $d(\partial)$. This premise needs correction, though it affects the final result in regard to the lower bound on l only in theories wherein a parity operator cannot be defined or the representation of the β-matrices is reducible or both. A brief discussion of the results is given in § 4.

† This bound is a straightforward consequence of the fact, evident from mere visual inspection of the spin-j block, that certain elements of the block are zero. This same fact could be represented by a graph (wherein an edge connecting a pair of vertices (k, l) is drawn if and only if the (k, l) element of the spin block is non-zero). The use of graph theory to obtain bounds on the rank has been advocated by Cox (1978, 1981).

‡ A suggestion that this premise may be suspect appears in the work of Loide and Loide (1977), where it is also shown that $l < 2s - 1$ in certain classes of theories.

2. Considerations on the lower bound on l

In the work leading to their bounds on l, Umezawa and Visconti observed that associated with a unique-mass equation of the form (1), there must exist a 'Klein–Gordon divisor' $d(\partial)$ defined by

$$d(\partial)(\mathrm{i}\beta^{\mu}\,\partial_{\mu}-m)=(\mathrm{i}\beta^{\mu}\,\partial_{\mu}-m)d(\partial)=-(\partial_{\mu}\,\partial^{\mu}+m^{2}). \tag{6}$$

From this defining equation it followed that

$$d(\partial)=m+\mathrm{i}\beta\cdot\partial+\sum_{p\geq 2}[(\mathrm{i}\beta\cdot\partial)^{p}+(\mathrm{i}\beta\cdot\partial)^{p-2}\,\partial^{2}]m^{-p+1} \tag{7a}$$

or

$$d(\partial)=m+\mathrm{i}\beta^{\mu}\,\partial_{\mu}+\sum_{p\geq 2}\mathrm{i}^{p}m^{-p+1}d^{\mu_1\cdots\mu_p}\,\partial_{\mu_1}\ldots\partial_{\mu_p} \tag{7b}$$

where

$$d^{\mu_1\cdots\mu_p}=\sum_{\mathscr{P}}(\beta^{\mu_1}\beta^{\mu_2}-g^{\mu_1\mu_2})\beta^{\mu_3}\ldots\beta^{\mu_p} \tag{8a}$$

$$=(d^{\mu_1\cdots\mu_{p-1}}\beta^{\mu_p})_{\mathrm{S}}. \tag{8b}$$

The subscript S in the last expression denotes that the product within the brackets is to be symmetrised in all the indices. It was required that the number of terms in the sum in (7) be finite in order that $d(\partial)$ be a local operator. This meant that if the highest value of p involved in the sum in (7) were denoted by $(l+1)$, one would have

$$d^{\mu_1\cdots\mu_{l+2}}=0 \tag{9}$$

and hence also $d^{\mu_1\cdots\mu_p}=0$ for all $p>(l+2)$. Equation (9) is just the condition (3).

The UV argument for the lower bound $(2s-1)$ on l then went as follows. The commutator/anticommutator of $\psi_\alpha(x)$ and $\psi_\beta(x')$ is given by $d_{\alpha\beta}(\partial)$ acting on a scalar function of $(x-x')$, and hence $d_{\alpha\beta}(\partial)$ transforms like $\psi_\alpha\times\psi_\beta$. Since $\psi_\alpha(x)$ describes the field with maximum spin s, the representation of $d_{\alpha\beta}(\partial)$ can then be decomposed into the sum of representations of spin $2s$, $2s-1,\ldots,0$, according to the Clebesh–Gordan theorem. This means in turn that $d(\partial)$ should contain matrix tensors of rank up to $2s$, i.e. $d^{\mu_1\cdots\mu_{2s}}\neq 0$. Therefore $d^{\mu_1\cdots\mu_p}$ can vanish only for $p>2s$ and hence $(l+2)>2s$ or $l\geq(2s-1)$.

The lower bound so deduced is however not honoured by all unique-mass relativistic equations (Mathews *et al* 1980), as already noted. The Hurley equation, involving a wavefunction transforming according to $(s,0)\oplus(s-\frac{1}{2},\frac{1}{2})$ is a case in point. So also is the 'doubled' Hurley equation in which the wavefunction has additional parts transforming as $(0,s)\oplus(\frac{1}{2},s-\frac{1}{2})$. While a hermitising operator η, defined to be such that

$$\eta\beta^{\mu}\eta^{-1}=\beta^{\mu+}, \tag{10}$$

does not exist in the simple Hurley equation, the doubled equation does admit an η and can be derived from a Lagrangian†. In both cases, the degree of β^0 is 3 (i.e.

† This equation is obviously reducible under the proper Lorentz group. One could add further Lorentz IR and produce an irreducible equation, but in this process the minimal degree would of course be increased. The discussion by Cox (1981) excludes reducible equations.

$l=1$) irrespective of the spin s, and the Klein–Gordon divisor is

$$d(\partial)=(\mathrm{i}\beta\cdot\partial+m)+(1/m)[\partial^2+(\mathrm{i}\beta\cdot\partial)^2]. \tag{11}$$

The violation of the UV bound in the former case could be attributed to the absence of η (Cox 1981), noting that the use of the commutation rules in the UV proof presupposes the existence of η. But such an explanation would be misleading insofar as it tends to create the impression—which will be seen below to be unjustified—that η has an essential role in the determination of bounds. Further, in the case of the 'doubled' Hurley equation which does possess η, one would still have to identify precisely what aspect of the UV proof is rendered invalid by the reducible nature of the equation.

It is readily apparent that the crucial element which plays a direct role in placing a lower bound on l is the transformation property of $d(\partial)$. A little reflection shows however that in presuming that $d(\partial)$ transforms like $\psi\times\psi$ (Umezawa 1956, Umezawa and Visconti 1956), there has been an oversight. Given any set of finite-dimensional matrices β^μ transforming as a four-vector and acting on a vector space which carries the representation $S(\Lambda)$ of the Lorentz group, i.e. given that

$$S(\Lambda)^{-1}\beta^\mu S(\Lambda)=\Lambda^\mu{}_\nu\beta^\nu, \tag{12}$$

the Lorentz IR involved in the transformation properties of all possible polynomial functions of the β^μ are all contained in $S\times S$, in view of a theorem of Gårding (1944) which will be stated in the next section. Whether or not the direct sum of these IR is equivalent to $S\times S$ will depend upon whether the set of β-matrices is irreducible or not; but in any case, *only a subset of these IR is involved in the transformation property* of $d(\partial)$. This fact ought to have been obvious, although, surprisingly, it had not been noted hitherto. The point is that only the matrices $d^{\mu_1\cdots\mu_p}$ which are *symmetric* tensors constructed from the β^μ occur in $d(\partial)$. These transform of course according to self-conjugate IR of the type (r, r) of the proper Lorentz group (or direct sums of such IR). Therefore any irreducible representations (m, n) with $m\neq n$ which are present in $S\times S$ *do not find a place* in the transformation of $d(\partial)$.

The implications of this fact for the question under consideration are strikingly brought out by the example of the simple Hurley equation based on the representation

$$S(\Lambda)\sim(s,0)\oplus(s-\tfrac{1}{2},\tfrac{1}{2}).$$

We have

$$S(\Lambda)\otimes S(\Lambda)\sim\sum_{j=0}^{2s}(j,0)\oplus\sum_{j=0}^{2s-1}[(j,1)\oplus(j,0)]\oplus\sum_{j=1}^{2s}(j-\tfrac{1}{2},\tfrac{1}{2}). \tag{13}$$

This representation of course contains spins up to $2s$ through the IR $(2s,0)$, $(2s-1,1)$ and $(2s-\frac{1}{2},\frac{1}{2})$. But the spin value $2s$ is not contained in any of the *self-conjugate* IR occurring in $S\times S$, which are just $(0,0)$, $(\frac{1}{2},\frac{1}{2})$ and $(1,1)$; only these are involved in the transformation of $d(\partial)$ and they contain only spins up to 2. The remainder of the UV argument then requires only that the degree $(l+2)$ of the minimal equation be $\geqslant 3$ instead of $(2s+1)$; there is no longer any conflict with the actual value of the minimal degree, which is 3.

The point raised above is not the only one to be considered in dealing with the general case. A full discussion will be possible on the basis of a fundamental theorem due to Gårding which will now be stated.

3. Gårding's theorem and its consequences

Gårding (1944) proved a theorem concerning the existence of sets of matrices T_α which transform as irreducible tensors under the action of a group in the following sense:

$$S_1(\Lambda)^{-1}T_\alpha S_2(\Lambda)=\sum_\beta D_{\alpha\beta}(\Lambda)T_\beta \tag{14}$$

where $S_1(\Lambda)$ and $S_2(\Lambda)$ are representation matrices belonging to two finite-dimensional representations of the group and $D(\Lambda)$ is an irreducible representation (see also Wightman 1978, Glass 1971b). The theorem may be stated as follows, in the context of the Lorentz group.

Theorem 1. (Gårding) The irreducible representations $D^{(\tau)}$ corresponding to which (non-vanishing) matrix tensor sets $T_\alpha^{(\tau)}$ can be defined are such that their direct sum is equivalent to $S_1 \otimes S_2$.

This theorem has been applied by Glass (1971b) to an analysis of the tensor basis of the algebra $\mathscr{A}(\beta)$ generated by the matrices β^μ which form a four-vector with respect to the Lorentz group $\mathscr{L}$ and are defined over a vector space which carries the representation $S(\Lambda)$ of $\mathscr{L}$. The algebra $\mathscr{A}(\beta)$ is constituted by products $\beta^{\mu_1}\beta^{\mu_2}\ldots\beta^{\mu_n}$ (for all n) and all finite linear combinations of such products. By a process of symmetrisation/antisymmetrisation over subsets of indices in products of the above type, followed by removal of traces, various matrix sets $B_\alpha^{(\tau)}$ transforming as irreducible tensors can be formed:

$$S(\Lambda)^{-1}B_\alpha^{(\tau)}S(\Lambda)=\sum_\beta D_{\alpha\beta}^{(\tau)}(\Lambda)B_\beta^{(\tau)}. \tag{15}$$

The algebra is spanned by such tensor sets. The basic question is as to which IR $D^{(\tau)}$ occur in the transformation properties of these irreducible tensors. The answer to this question has been given by Glass (1971b) as follows.

Theorem 2. (a) If the algebra $\mathscr{A}(\beta)$ is irreducible, there exists a basis of $\mathscr{A}(\beta)$ consisting of irreducible tensor sets; the tensor sets exist in one-to-one correspondence with the IR of $\mathscr{L}$ whose direct sum is equivalent to $S(\Lambda)\otimes S(\Lambda)$. (b) If the matrices constituting the algebra $\mathscr{A}(\beta)$ form a reducible set, acting irreducibly on and leaving invariant q vector spaces of the representations $S_1(\Lambda), S_2(\Lambda), \ldots, S_q(\Lambda)$ of $\mathscr{L}$, then irreducible tensor sets of matrices exist which together span $\mathscr{A}(\beta)$, and they can be placed in one-to-one correspondence with the IR of $\mathscr{L}$ whose direct sum is equivalent to $(S_1\otimes S_1)\oplus(S_2\otimes S_2)\oplus\ldots\oplus(S_q\otimes S_q)$.

The implications of the above theorems for the question under consideration are readily apparent. Considering all the *self-conjugate* IR (r, r) contained in $S\otimes S$ or $\Sigma\, S_i\otimes S_i$ according as the algebra $\mathscr{A}(\beta)$ is irreducible or reducible, let k be the highest value of r which occurs. Then the algebra contains a non-vanishing tensor of rank $2k$, totally symmetric in the β-matrices. This means that $d^{\mu_1\cdots\mu_{2k}}$ cannot vanish. None of the $d^{\mu_1\cdots\mu_p}$ for any $p<2k$ can vanish either: if one did, all higher ones would vanish too since $d^{\mu_1 \cdots \mu_{p+1}}$ is a symmetrised product of $d^{\mu_1 \cdots \mu_p}$ and $\beta^{\mu_{p+1}}$. Therefore the lowest value possible for $l+2$ in equation (9) is $(2k+1)$; we thus have the following theorem†.

Theorem 3. If the highest-ranking self-conjugate IR contained in the decomposition of $S \otimes S$ (if $\mathscr{A}(\beta)$ is irreducible) or $\Sigma\, S_i \otimes S_i$ (if $\mathscr{A}(\beta)$ is fully reducible) is (k, k), the degree $(l+2)$ of the minimal equation of β^0 cannot be less than $(2k+1)$, or

$$l \geqslant (2k-1). \tag{16}$$

The value of k itself can easily be found from a knowledge of the IR contained in S (or each of the S_i, in the case of reducible $\mathscr{A}(\beta)$). Let $\tau \equiv (m, n)$ and $\tau' \equiv (m', n')$ be any two of the IR contained in one and the same S_i. The highest-ranking self-conjugate IR contained in the direct product of the above two IR—this direct product being itself contained in $S_i \times S_i$—is (r, r) with $r = \min(m+m', n+n')$. Let $r_{\max}$ be the highest of the values of r arising when all such direct products for all the S_i are considered. Then

$$k = r_{\max}. \tag{17}$$

The reason why the degree of the minimal equation in the case of the 'doubled' Hurley equation can be just 3 for any s, despite the existence of a hermitising operator η, can now be seen. It is just that the matrices β^μ in this case from a *reducible* set, the Lorentz representations associated with the reduced parts being $S_1 \sim (s, 0) \oplus (s-\frac{1}{2}, \frac{1}{2})$, and S_2, the conjugate of S_1. Both $S_1 \otimes S_1$ and $S_2 \otimes S_2$ contain only $(0, 0)$, $(\frac{1}{2}, \frac{1}{2})$ and $(1, 1)$ as self-conjugate IR. Hence $k = 1$, leading to the lower bound 3 on $(l+2)$—which is honoured, unlike the UV lower bound $(2s+1)$ which we have seen to be not really applicable. It is worth emphasising that $d(\partial)$ in this case does not transform like $\psi \times \psi$ even under the *rotation* group, which would be hard to understand on the basis of the UV arguments.

Reducible equations are undoubtedly not of great interest, and even among irreducible equations, it is those equations which admit a hermitising operator η which are of the greatest physical interest. What can one say about the value of k for this class of equations? The answer is that when both irreducibility of $\mathscr{A}(\beta)$ and existence of η are demanded, k is equal to the maximum spin present in S, namely j_m, and consequently

$$l \geqslant 2j_m - 1. \tag{18}$$

To see this, we observe first that in any theory admitting η, along with any IR, say (m, n), present in S, its conjugate (n, m) must also be present. Consider now a representation, say (m_0, n_0), which contains the highest spin j_m. Among the self-conjugate IR contained in $S \times S$ there will be some arising from the direct product of (m_0, n_0) with its conjugate (n_0, m_0), of which that of the highest rank is evidently (m_0+n_0, m_0+n_0), i.e. (j_m, j_m). No IR (r, r) with $r > j_m$ can arise from the direct product of any other pair of IR (m, n), (m', n'). For, the highest r that one can get from the product of (m, n) and (m', n') is $\min(m+m', n+n')$, and if this is to exceed (m_0+n_0) we would have to have $m+m' > m_0+n_0$, $n'+n > m_0+n_0$ and hence $m+n+m'+n' >$

† In the case of multimass equations, one has $\beta_0^l\, \Pi_i(\mu_i^2 \beta_0^2 - 1) = 0$ instead of (2) where the μ_i are mass ratios. Consequently $(l+2)$ is to be replaced by $(l+2N)$ in the statement of the theorem, N being the number of distinct masses.

$2(m_0+n_0)$. But this is not possible since, by the definition of (m_0, n_0), there is no IR in S for which $m+n>m_0+n_0$. This concludes the proof that $k=j_m$ and hence that equation (18) is valid under the conditions stated.

4. Discussion

The main points brought out in the foregoing considerations on the lower bound on l are: (a) that any analysis of the transformation property of $d(\partial)$ must be done in terms of Lorentz IR (rather than in terms of IR of the rotation subgroup as had been done hitherto), and that explicit account must be taken of the fact that $d(\partial)$ contains only tensors which are totally symmetric; (b) that the question of reducibility of $\mathscr{A}(\beta)$ is crucial for the determination of the irreducible tensors involved, given $S(\Lambda)$; and (c) that the condition of existence of η is by itself of no direct relevance to the problem, *there being a uniform method* to determine the minimal degree of β^0 irrespective of whether η exists or not. However, this condition, taken together with the irreducibility of $\mathscr{A}(\beta)$, requires that $l \geqslant 2j_m - 1$, a result is proved here for the first time. The conjectured bound $l \geqslant 2s-1$ for which a predilection is shown in earlier works (Chandrasekharan *et al* 1972, Cox 1981) is weaker than the proven bound in which j_m (and not s) figures. The conjecture seems to have been prompted by the idea that the algebra of the β-matrices 'must provide at least the physical degrees of freedom in the theory, which simply amount to $2s+1$' (Cox 1981). The fact that in theories such as that of the doubled Hurley equation with η, wherein $(2s+1)$ degrees of freedom exist for arbitrary s despite the minimal degree being just 3, should be enough to discount the idea. However, the alternative argument of Cox, in terms of the minimum number of linked IR which must exist for given s in an irreducible theory with η, is plausible.

The matrices β^μ in barnacled theories (Hurley and Sudarshan 1975, Khalil 1978) form a reducible but indecomposable set, of a special type. The tensor content of indecomposable algebras $\mathscr{A}(\beta)$ is not completely defined, unlike in the irreducible and fully reducible cases, as it is possible only to say that the IR according to which the tensors transform constitute a proper subset of the set of IR contained in $S \otimes S$ (Glass 1971b). The question of fixing a lower bound on l in such theories requires further consideration.

A remark about the UV upper bound may not be out of place here, especially in view of an apparent misconception about the reason for its breakdown. Cox (1981), for instance, has claimed that it is not true in general that $d(\partial)$ is a polynomial in $(i\beta \cdot \partial)$ and inferred that it is because the UV proof takes $d(\partial)$ to be a polynomial that it fails to be generally valid. Actually this is not the case: the polynomial form (7) of $d(\partial)$ is completely general. Cox's claim seems to be based on a misunderstanding of the following point made by Glass. Under the circumstance that the maximum spin present in $d(\partial)$ is $2s$, Umezawa and Visconti had argued that the series in (7) should terminate with $p=2s$ (as $d^{\mu_1\cdots\mu_p}$ with $p>2s$ would involve spin $>2s$). Glass pointed out that $d^{\mu_1\cdots\mu_p}$ is a reducible tensor, and hence that even for $p>2s$ it need not vanish *in toto* in order to be free of spin $>2s$; only its reduced parts involving spins $>2s$ need vanish. In particular, $d^{\mu_1\cdots\mu_{2s+1}}$ need not vanish though its traceless part $\bar{d}^{\mu_1\cdots\mu_{2s+1}}$ must vanish. The vanishing of $\bar{d}$ by itself (without d as a whole vanishing) does not lead to an equation of the form (2) or (3) since $\bar{d}$ involves contracted pairs of β-matrices. In such a case it is only some higher-rank $d^{\mu_1\cdots\mu_p}$ $(q>2s+1)$ which would vanish as a whole, thus making $l+2>2s+1$. Though in such a case the vanishing of $\bar{d}$ would permit the re-expression of the terms with $p>2s$ in (7*a*) in a form in which β appears also otherwise than in the combination $(\beta\cdot\partial)$, it does not alter the fact that it is only an alternative form for the polynomial expression (7*a*). The reducibility to a non-polynomial form is a consequence rather than the cause of the breakdown of the UV bound. The reason for the breakdown of the upper bound in theories involving repeated appearance of Lorentz IR is to be found in the fact that with increase in multiplicity, the number of tensors of the type (r, r) present in $S\otimes S$ also increases, and these cannot all be constructed from products of up to $2s$ factors of β^{μ} only. This necessitates the existence of non-vanishing tensors of higher rank than $2s$ and, correspondingly, a higher degree for the minimal equation.

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Note added in Proof: W Cox (1982 *J. Phys. A: Math. Gen.* **15** 223) has considered a number of irreducible, parity-invariant spin-2 theories. Our theorem, which requires that $1\geq 2j_m-1$ for these, enables the bounds set by Cox on l (q in his notation) to be tightened to $l=5$, $5\leq 1\leq 7$ and $l=5$ respectively for the theories of §§ 4, 5, and 6 of his paper.

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